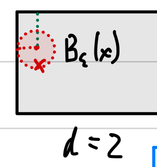
Math 451: Introduction to General Topology
Lecture 6

Examples (continued).

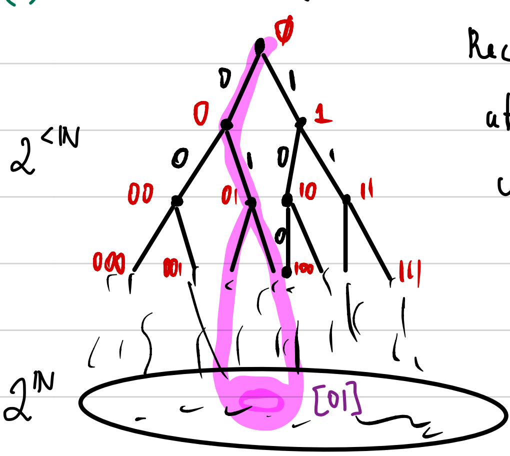
(b) In $(\mathbb{R}^d, d_2)$ all open rectangles are open sets, where an **open rectangle** is a set of the form $I_1 \times I_2 \times \dots \times I_d$, where each $I_k \subseteq \mathbb{R}$ is an open interval.

Proof. For $x \in R := I_1 \times I_2 \times \dots \times I_d$, we see that each coordinate $x_k \in I_k$, let $\varepsilon_k > 0$ be such that $(x_k - \varepsilon_k, x_k + \varepsilon_k) \subseteq I_k$. Set $\varepsilon := \min_{1 \le k \le d} \varepsilon_k$, then $B_\varepsilon(x) \subseteq R$.



QED

(c) Let Σ be a ctbl nonempty set and $X := \Sigma^{\mathbb{N}}$ with the "realtors' metric, defined previously.



Recall that every cylinder $[w]$, for $w \in \Sigma^{<\mathbb{N}}$, is an open ball at any $x \in \Sigma^{\mathbb{N}}$ that extends w (write $x \succcurlyeq w$), and also a closed ball at x. In particular, $[w]$ is open.

Claim. $[w]$ is also closed.

Proof. Let $n := lh(w)$. Then $[w]^c = \bigcup_{\substack{v \in \Sigma^n \\ v \neq w}} [v]$ hence $[w]^c$ is open being a union of open sets. Hence $[w]$ is closed. QED

Thus every cylinder is **clopen**, i.e. closed and open.

Obs. For any cylinders $[w]$ and $[v]$, they are either disjoint or one is contained in the other.

Proof. If the word $w \succcurlyeq v$ then $[w] \subseteq [v]$. If $w \not\succcurlyeq v$ and $v \not\succcurlyeq w$, then $\exists\, i < lh(w), lh(v)$ with $w(i) \neq v(i)$, hence $[w] \cap [v] = \emptyset$. $\square$

Prop. Every open set in $\Sigma^{\mathbb{N}}$ is a ctbl union of disjoint cylinders.

Proof. Let $U \subseteq \Sigma^{\mathbb{N}}$ be open. Take $W := \{w \in \Sigma^{<\mathbb{N}} : [w] \subseteq U \text{ and } \nexists \text{ shorter } v \preccurlyeq w \text{ with } [v] \subseteq U\}$. Clearly for any two distinct $w_1, w_2 \in W$, we have $w_1 \not\succcurlyeq w_2$ and $w_2 \not\succcurlyeq w_1$ so $[w_1] \cap [w_2] = \emptyset$. Thus it suffices to show that $U = \bigcup_{w \in W} [w]$

because $W \subseteq \Sigma^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} \Sigma^n$ is ctbl. By def. $\bigcup_{w\in W}[w] \subseteq \mathcal{U}$. To show the converse, fix $x \in \mathcal{U}$. Then $\exists$ open ball, namely a cylinder $[v]$, s.t. $x \in [v] \subseteq \mathcal{U}$. Let $u \prec v$ be the shortest s.t. $[u] \subseteq \mathcal{U}$. Then $u \in W$, so $x \in \bigcup_{w\in W}[w]$. $\boxed{\text{QED}}$

Prop. Let (X,d) be a metric space and $Y \subseteq X$. Then the open sets in the subspace (Y,d) are exactly sets of the form $\mathcal{U} \cap Y$ where $\mathcal{U} \subseteq X$ is an open set in (X,d).

Proof. HW.

Examples.

Let $X := \mathbb{R}$ with the usual metric d.

(a) If $Y = [0,1]$, then $(\frac{1}{2},1]$ is open in the subspace (Y,d), in fact it is the open ball at 1 of radius $\frac{1}{2}$.

(b) If $Y = [0,1] \cup \{2\}$, then $\{2\}$ is open in the subspace (Y,d), in fact $\{2\} = B^Y_{\frac{1}{2}}(2)$.

$\{2\}$ is closed in Y because $Y \setminus \{2\} = [0,1]$ is open since $[0,1] = (-\frac{1}{2},\frac{3}{2}) \cap Y$.

Limits of sequences in metric spaces.

Recall that a sequence (x_n) is just a function with domain $\mathbb{N}$.

Def. For a subset $P \subseteq \mathbb{N}$, we write:

- $\forall^\infty n \in \mathbb{N}\ n \in P$ if all but finitely many $n \in \mathbb{N}$ are in P.
  Equivalently, $\exists k \in \mathbb{N}\ \forall n \geq k\ n \in P$, i.e. eventually $x_n \in P$.

  Thus, $\forall^\infty_n = \exists k \in \mathbb{N}\ \forall n \geq k$.
- $\exists^\infty n \in \mathbb{N}\ n \in P$ if there are infinitely many n in P, i.e. P is infinite.
  Equivalently, $\forall k \in \mathbb{N}\ \exists n \geq k\ n \in P$.

  Thus, $\exists^\infty n = \forall k \in \mathbb{N}\ \exists n \geq k$.

Obs. $\left(\text{not } \forall^\infty_{n\in\mathbb{N}}\ n\in P\right) = \exists^\infty_{n\in\mathbb{N}}\ n\notin P.$

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Def. Let (X,d) be a metric space, $(x_n)\subseteq X$ be a sequence, and $x\in X$. We say that (x_n) *converges* to x, and write $\lim_{n\to\infty} x_n = x$, if $\forall \varepsilon>0\ \underbrace{\exists N\in\mathbb{N}\ \forall n\geq N}_{\forall^\infty_n}\ \underbrace{d(x,x_n)<\varepsilon}_{x_n\in B_\varepsilon(x)}$.

In other words, for every open ball B at x, for all but finitely many n, $x_n\in B$.

Prop. Let (X,d) be a metric space, $(x_n)\subseteq X$ be a sequence, and $x\in X$. TFAE:

(1) $\lim_{n\to\infty} x_n = x$, i.e. for every open ball at x, $\forall^\infty_n\ x_n\in B$.

(2) $\forall$ open $U\ni x$, $\forall^\infty_n\ x_n\in U$. This is the "correct" definition of convergence.

(3) $\lim_{n\to\infty} d(x_n,x) = 0$.

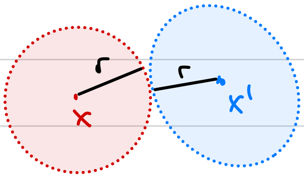
Proof. (1) $\Leftrightarrow$ (3) is a chasing-definitions proof. HW

(2) $\Rightarrow$ (1) is trivial since open balls are open sets.

(1) $\Rightarrow$ (2). Let $U\ni x$ be open. Then there is a ball B at x with $B\subseteq U$. But then by (1), $\forall^\infty_n\ x_n\in B\subseteq U$. $\boxed{\text{QED}}$

Prop (uniqueness of limit). Let (X,d) be a metric space and $(x_n)\subseteq X$ be a sequence. Then the limit of (x_n), if exists, is unique, i.e. for all $x,x'\in X$ if $\lim_n x_n = x$ and $\lim_n x_n = x'$ then $x=x'$.

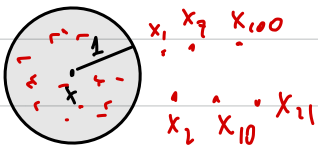
Proof. Let $x\neq x'$ and suppose that $\lim_{n\to\infty} x_n = x$. We show that $\lim_{n\to\infty} x_n \neq x'$.



Since $x\neq x'$, $d(x,x')>0$ so take $r := \frac{1}{2}d(x,x')$. Then $B_r(x)\cap B_r(x') = \emptyset$. But $\forall^\infty_n\ x_n\in B_r(x)$ hence $\forall^\infty_n\ x_n\notin B_r(x')$, in particular, not $\forall^\infty_n\ x\in B_r(x')$. $\boxed{\text{QED}}$

Obs. In a metric space (X,d), if a sequence (x_n) converges then it is **bounded**, i.e. $\{x_n\}$ is contained in some ball.

Proof. Let (x_n) converge to some $x \in X$. Then $\forall^\infty_n \; x_n \in B_1(x)$.



Let $Y := \{x_n : x_n \notin B_1(x)\}$, so Y is finite.

Let $r := \max_{y \in Y} d(x,y)$. Then $Y \subseteq B_{r+1}(x)$ but also $\{x_n : x_n \notin Y\} \subseteq B_1(x) \subseteq B_{r+1}(x)$,

so $\{x_n\} \subseteq B_{r+1}(x)$. $\boxed{\text{QED}}$

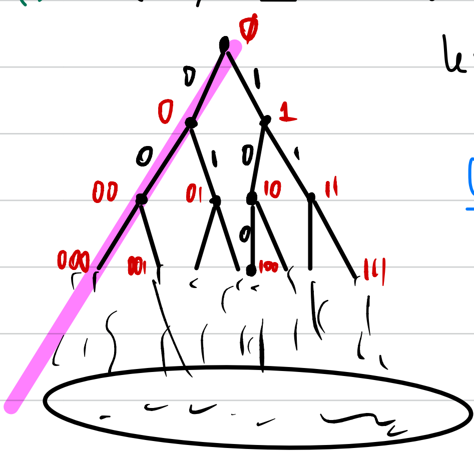
Examples.

(a) Let X be set and d be 0-1 metric on it, i.e. $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$

For a sequence (x_n) and $x \in X$,

$$\lim_n x_n = x \iff \forall^\infty_n \; x_n = x.$$

(b) Let $X = \Sigma^{\mathbb{N}}$ for some ctbl nonempty Σ, with the usual metric d.



Let $x_n := 0^n 1^\infty = \underbrace{00\ldots0}_{n}111\ldots$

Claim. $\lim_n x_n = 0^\infty$.

Proof. Indeed, every ball at 0^∞ is of the form $[0^k]$ for some $k \in \mathbb{N}$ and $\forall n \geq k$, $x_n = 0^n 111\ldots \in [0^k]$. $\boxed{\text{QED}}$

More generally, the following is true:

Prop. A sequence $(x_n) \in \Sigma^{\mathbb{N}}$ converges to $x \in \Sigma^{\mathbb{N}} \iff$ for each index $i \in \mathbb{N}$ $\forall^\infty_n \; x_n(i) = x(i)$.

Proof. HW